# ON THE STABILITY OF MOTION OF A GYROSCOPE ON GIMBALS, II

**(OB USTOICHIVOSTI DVIZHENIIA GIROSKOPA V KARDANOVOM PODVESE II)**

*PMM Vol.22, No.4, 1958, pp. 499-503*

V.V. RUMIANTSEV
(Moscow)

*(Received 11 April 1958)*

The author investigates the stability of a certain kind of motion of a symmetrical gyroscope on gimbals when the stationary axis of the outer gimbal ring is horizontal. This note is a continuation of [1].

1. Let the fixed point $O$ of a symmetrical gyroscope coincide with the origin of a fixed, rectangular, coordinate system $O\xi\eta\zeta$; the $O\zeta$ is horizontal and coincides with the fixed axis of rotation of the outer gimbal ring. The rectangular coordinate system $Oxyz$ moves with the inner gimbal ring; the axis $Ox$ coincides with the axis of rotation of the inner ring, the axis $Oz$ coincides with the axis of the gyroscope. The directions of the axes $O\eta$ and $Oy$ are such that the rectangular coordinate systems $O\xi\eta\zeta$ and $Oxyz$ are right-handed. Let the axes $x$, $y$ and $z$ coincide with the principal axes of inertia of the inner ring with respect to the point $O$; let the weight of the gyroscope plus the inner ring be $P$ and let the center of gravity of the gyroscope and the inner ring be on the $z$-axis, its coordinates being $(O,\ O,\ z_0)$. Let $A = B$, $C$ and $A_1$, $B_1$, $C_1$ be principal moments of inertia of the gyroscope and of the inner ring with respect to the fixed point $O$ respectively and let $A_2$ be the moment of inertia of the outer ring with respect to its axis $O\zeta$.

As the independent generalized coordinates, which define the orientation of the mechanical system under consideration in the space $O\xi\eta\zeta$, we shall take the Eulerian angles: the angle of nutation $\theta$ (between the axes $\zeta$ and $z$), the angle of precession $\psi$ (between the axes $\xi$ and $x$) and the angle of rotation of the gyroscope itself $\phi$, that is the angle through which the gyroscope turned about the axis $Oz$ with respect to the inner gimbal ring. Projections of the instantaneous angular velocity of the gyroscope $\omega$, and of the inner ring $\omega_1$, on the moving coordinate axes are expressed by the following formulas:

$$p = \theta', \qquad q = \psi' \sin\theta, \qquad r = \varphi' + \psi' \cos\theta$$
$$p_1 = \theta', \qquad q_1 = \psi' \sin\theta, \qquad r_1 = \psi' \cos\theta \tag{1.1}$$

The vector of the instantaneous angular velocity of the outer ring is directed along the $O\zeta$ axis; its projection on the $O\zeta$ axis equals $\psi'$.

We shall use the following expression for the kinetic energy of the system (1.1,1)

$$T = \tfrac{1}{2}\{(A + A_1)\theta'^2 + [(A + B_1)\sin^2\theta + C_1\cos^2\theta + A_2]\psi'^2 + C(\varphi' + \psi'\cos\theta)^2\}$$

and the following force function of the gravity forces

$$U = -Pz_0 \sin\theta \sin\psi \tag{1.1,2}$$

Constructing the Lagrange function $L = T + U$, we obtain the following equations of motion for our system:

$$\begin{gathered}
(A + A_1)\theta'' - (A + B_1 - C_1)\psi'^2 \sin\theta\cos\theta + C(\varphi' + \psi'\cos\theta)\psi'\sin\theta + \\
+ Pz_0\cos\theta\sin\psi = 0 \\
\frac{d}{dt}\{[(A + B_1)\sin^2\theta + C_1\cos^2\theta + A_2]\psi' + C(\varphi' + \psi'\cos\theta)\cos\theta\} + \\
+ Pz_0 \sin\theta\cos\psi = 0 \\
C\frac{d}{dt}(\varphi' + \psi'\cos\theta) = 0
\end{gathered} \tag{1.2}$$

The equations of motion (1.2) admit the first integrals

$$\begin{gathered}
(A + A_1)\theta'^2 + [(A + B_1)\sin^2\theta + C_1\cos^2\theta + A_2]\psi'^2 + \\
+ C(\varphi' + \psi'\cos\theta)^2 + 2Pz_0\sin\theta\sin\psi = h \\
\varphi' + \psi'\cos\theta = r = \text{const}
\end{gathered} \tag{1.3}$$

the first of which is the energy integral. We shall investigate the stability of rotation of the gyroscope about a vertical axis, described by the particular solution of the equations (1.2):

$$\theta = \tfrac{1}{2}\pi, \quad \theta' = 0, \quad \psi = \tfrac{1}{2}\pi, \quad \psi' = 0, \quad r = \omega \tag{1.4}$$

It is immediately seen that in the (1.4) case, the middle plane of the outer ring is horizontal and the middle plane of the inner ring is vertical.

Substituting perturbations

$$\theta = \tfrac{1}{2}\pi + \eta_1, \quad \theta' = \eta', \quad \psi = \tfrac{1}{2}\pi + \eta_2, \quad \psi' = \eta'_2, \quad r = \omega + \xi \tag{1.4,1}$$

we easily find that the equations of the perturbed motion admit the following first integrals:

$$V_1 = (A + A_1)\eta_1'^2 + (A + B_1 + A_2)\eta_2'^2 + C(\xi^2 + 2\omega\xi) - \\ - Pz_0(\eta_1^2 + \eta_2^2) + \ldots = \text{const}$$

$$V_2 = \xi = \text{const} \tag{1.5}$$

The first of the above integrals contains the first and the second order terms only with respect to the variables $\eta_1$ and $\eta_2$. The function:

$$V = V_1 - 2C\omega V_2 = (A + A_1)\eta_1'^2 + (A + B_1 + A_2)\eta_2'^2 + \\ + C\xi^2 - Pz_0(\eta_1^2 + \eta_2^2) + \cdots \tag{1.6}$$

is a positive-definite function of its variables only when

$$z_0 < 0 \tag{1.6,1}$$

Hence, the above inequality is a sufficient condition for stability of motion (1.4) with respect to the variables $\theta$, $\theta'$, $\psi$, $\psi'$, $r$ * for any value of $\omega$. It is easily seen that the stability is of a secular kind.

In the case of an equilibrated gyroscope ($z_0 = 0$) the motion (1.4) is stable with respect to the variables $\theta'$, $\psi'$, $r$.

When $z_0 > 0$, the degree of instability is even, hence the gyroscopic stabilization is possible according to Kelvin's theorem. We shall find the condition for the gyroscopic stabilization by utilizing the fundamental Chetaev theorem [2] on the existence of a definite quadratic integral of the variational equations for a stable unperturbed motion.

It is easily seen that in the considered case the variational equations

$$(A + A_1)\eta_1'' + C\omega\eta_2' - Pz_0\eta_1 = 0$$

$$(A + B_1 + A_2)\eta_2'' - C\omega\eta_1' - Pz_0\eta_2 = 0 \tag{1.7}$$

admit the integral [2]

$$\Gamma = 2[(A + A_1)\eta_1'\eta_2 - (A + B_1 + A_2)\eta_1\eta_2'] + C\omega(\eta_1^2 + \eta_2^2) - \\ - \frac{A_1 - B_1 - A_2}{2C\omega}[(A + B_1 + A_2)\eta_2'^2 - (A + A_1)\eta_1'^2 + \\ + Pz_0(\eta_1^2 - \eta_2^2)] = \text{const} \tag{1.8}$$

Let us consider the following function:

$$V = \tfrac{1}{2}C\omega V_1 + Pz_0\Gamma - C^2\omega^2 V_2 =$$

$$= \frac{C^2\omega^2 + (A_1 - B_1 - A_2)Pz_0}{2C\omega}(A + A_1)\eta_1'^2 + 2(A + A_1)Pz_0\eta_1'\eta_2 +$$

* Skimel V.N. Nekotorye Zadachi ob ustoichivosti dvishenia tverdego tela. (Certain problem of stability of motion of a rigid body) Avtoreferat dissertatsii, Kazan, 1955.

$$+\frac{C^2\omega^2+(A_1-B_1-A_2)Pz_0}{2C\omega}Pz_0\eta_2^2+\frac{1}{2}C^2\omega\zeta^2+$$
$$+\frac{C^2\omega^2+(A_2+B_1-A_1)Pz_0}{2C\omega}(A+B_1+A_2)\eta_2'^2-2(A+B_1+A_2)Pz_0\eta_1\eta_2'+$$
$$+\frac{C^2\omega^2+(A_2+B_1-A_1)Pz_0}{2C\omega}Pz_0\eta_1^2+\dots \qquad (1.9)$$

In the case when $z_0 > 0$, the function $V$ is a positive-definite function of its variables if the following single condition is satisfied:

$$C^2\omega^2-(2A+A_1+B_1+A_2+2\sqrt{(A+A_1)(A+B_1+A_2)})Pz_0>0 \qquad (1.10)$$

The condition (1.10) turns out to be the condition for the gyroscopic stabilization of the motion (1.4) with respect to the variables $\theta$, $\psi$, $\theta'$, $\psi'$, $r$.

Neglect of masses of the gimbal rings reduces (1.10) to the well-known Maievki condition $C^2\omega^2 - 4APz_0 > 0$, which is the necessary and sufficient condition for stability of rotation of the Lagrange gyroscope about a vertical axis. It could be proved* that the inequality (1.10) is also the necessary condition for the stability of motion (1.4).

If the gyroscopic stabilization does take place, it will be sooner or later destroyed by the dissipative forces; that is, the stability of motion (1.4) in the case $z_0 > 0$ and satisfying the condition (1.10) is temporary.

In order to prove the above statement, we shall assume that in the perturbed motions the dissipative forces which are derivatives of the positive-definite Raleigh function are present.

$$2f=a\eta_1'^2+2b\eta_1'\eta_2'+c\eta_2'^2 \qquad (1.1A)$$

The approximate equations of the perturbed motion are

$$(A+A_1)\eta_1''+C\omega\eta_2'-Pz_0\eta_1=-a\eta_1'-b\eta_2'$$
$$(A+B_1+A_2)\eta_2''-C\omega\eta_1'-Pz_0\eta_2=-b\eta_1'-c\eta_2' \qquad (1.11)$$

Let us consider the function

$$2W=(A+A_1)\eta_1'^2+(A+B_1+A_2)\eta_2'^2-Pz_0(\eta_1^2+\eta_2^2)-$$
$$-4\varepsilon[(A+A_1)\eta_1\eta_1'+(A+B_1+A_2)\eta_2\eta_2'] \qquad (1.12)$$

and its time derivative, in view of the differential equations (1.11),

$$W'=-\{[a+2\varepsilon(A+A_1)]\eta_1'^2+2b\eta_1'\eta_2'+[c+2\varepsilon(A+B_1+A_2)]\eta_2'^2+$$
$$+2Pz_0\varepsilon(\eta_1^2+\eta_2^2)-2\varepsilon[C\omega(\eta_1\eta_2'-\eta_1'\eta_2)+b(\eta_1\eta_2'+\eta_1'\eta_2)+$$
$$+a\eta_1'\eta_1+c\eta_2\eta_2']\} \qquad (1.13)$$

* See the preceding footnote.

Here $\epsilon$ is a positive constant, sufficiently small to make the main diagonal minors of the discriminant of the quadratic form $W'$ positive. Then, the function $W'$ will be negative-definite, the function $W$ will have an infinitely small upper bound and also could be made negative by a suitable choice of numerically small values of $\eta_1$, $\eta_1'$, $\eta_2$, $\eta_2'$. On the strength of the Liapunov theorem on instability we conclude that the motion (1.4) is unstable with respect to the variables $\theta$, $\psi$, $\theta'$, $\psi'$, when dissipative forces are present.

We shall consider also the problem of the rotational stability about a horizontal axis of a heavy symmetrical gyroscope, as defined by the particular solution of the equations (1.2):

$$\theta = 0, \quad \theta' = 0, \quad \psi = 0, \quad \psi' = 0, \quad r = \omega \tag{1.14}$$

In this case the middle surfaces of the inner and outer rings are vertical and coincide with each other. We shall prove that the motion (1.14) is unstable with respect to the variables $\theta$, $\psi$, $\theta'$, $\psi'$. The equations of the perturbed motion in this case have the form

$$\begin{gathered}(A + A_1)\eta_1'' - (A + B_1 - C_1)\eta_1\eta_2'^2 + C(\omega + \zeta)\eta_1\eta_2' + \\ + Pz_0\eta_2\left(1 - \tfrac{1}{2}\eta_1^2\right) + \ldots = 0\end{gathered}$$

$$\begin{gathered}(A_2 + C_1)\eta_2'' + 2(A + B_1 - C_1)\eta_1\eta_1'\eta_2' + (A + B_1 - C_1)\eta_1^2\eta_2'' + \\ + Pz_0\eta_1\left(1 - \tfrac{1}{2}\eta_2^2\right) - C(\omega + \zeta)\eta_1\eta_1' + \ldots = 0\end{gathered} \tag{1.15}$$

where the rows of dots indicate omitted terms of higher order than three. Let us consider the function

$$V = (A + A_1)\eta_1'\eta_2 + (A_2 + C_1)\eta_1\eta_2' \tag{1.16}$$

whose derivative $V'$, in view of (1.15), with terms above the second order omitted, is

$$V' = -Pz_0(\eta_1^2 + \eta_2^2) + (A + A_1 + A_2 + C_1)\eta_1'\eta_2' + \ldots \tag{1.16,1}$$

Let, for example, $z_0 > 0$. In one of the parts of the region $V < 0$, defined by the simultaneous inequalities

$$\eta_1 < 0, \quad \eta_2 > 0, \quad \eta_1' < 0, \quad \eta_2' > 0 \tag{1.16,2}$$

the function $V'$ is a negative-definite function for sufficiently small numerical values of the variables.

It follows that the function $V$ satisfies all conditions of the Chetaev theorem on instability, which proves our previous statement that the motion defined by (1.14) is unstable.

2. We shall now consider an interesting case of a gyroscope on gimbals with a moment of external forces $L_z$ applied to the axis $Oz$. In particular, we could choose a moment such that the angular velocity of rotation of

the gyroscope itself would remain constant; that is $\phi'$ = constant. Let us assume that the moment $L_z$ is a continuous function of the Eulerian angles and their time derivatives.

In the case under consideration, the equations of motion of a symmetrical heavy gyroscope with a horizontal axis of rotation of the outer gimbal ring could be reduced to the form

$$(A + A_1)\theta'' - (A + B_1 - C_1)\psi'^2 \sin\theta\cos\theta + Cr\psi'\sin\theta + Pz_0\cos\theta\sin\psi = 0$$
$$\frac{d}{dt}\{[(A + B_1)\sin^2\theta + C_1\cos^2\theta + A_2]\psi'\} - Cr\theta'\sin\theta + Pz_0\sin\theta\cos\psi = 0 \tag{2.1}$$
$$C\frac{dr}{dt} = L_z$$

Irrespective of the character of the function satisfying the properties of $L_z$ the first integral of the equations of motion (2.1) is

$$(A + A_1)\theta'^2 + [(A + B_1)\sin^2\theta + C_1\cos^2\theta + A_2]\psi'^2 + 2Pz_0\sin\theta\sin\psi = \text{const} \tag{2.2}$$

and it is analogous to the energy integral.

We shall assume also that the equations (2.1) admit particular solutions of the kind (1.4) or (1.14), which is obviously possible when $L = 0$. It is easily seen that all the reasoning given in Section 1 concerning the stability or instability of motions described by the particular solutions of (1.4), or (1.14) is valid for the present case.

Indeed, the equations of the perturbed motion (1.4) have the first integral

$$V_1 = (A + A_1)\eta_1'^2 + (A + B_1 + A_2)\eta_2'^2 - Pz_0(\eta_1^2 + \eta_2^2) + \ldots = \text{const} \tag{2.3}$$

which at $z_0 < 0$ is positive-definite with respect to the variables $\eta_1$, $\eta_2$, $\eta_1'$, $\eta_2'$, showing that at $z_0 < 0$ the unperturbed motion (1.4) is stable with respect to the variables $\theta$, $\theta'$, $\psi$, $\psi'$.

As before, the variational equations are of the form (1.7), and admit the integral (1.8). Examining the Liapunov function

$$V = \frac{1}{2}C\omega V_1 + Pz_0\Gamma \tag{2.3,1}$$

we obtain for $z_0 > 0$ the condition (1.10) for the gyroscopic stabilization (first approximation) of the motion (1.4) being destroyed by dissipative forces.

Examination of the function (1.16) shows that in this case the motion (1.14) is unstable.

## BIBLIOGRAPHY

1. Rumiantsev, V.V., Ob ustoichivosti giroskopa v kardanovom podvese (On the stability of motion of a gyroscope on gimbals). *PMM* Vol. 22, No. 3, 1958.

2. Chetaev, N.G., O nekotorykh zadachakh ob ustoichivosti dvizheniia v mekhanike (On certain problems of stability of motion in mechanics). *PMM* Vol. 20, No. 3, 1956.

3. Magnus, K., Beiträge zur Dynamik des kraftfreien, kardanisch gelagerten Kreisels. *Z. Angew. Math. Mech.* Bd. 35, No. 1/2, 1955.


*Translated by* T.L.